Roots and Fruits of Decoherence

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Abstract. The concept of decoherence is defined and discussed in a historical context. This is illustrated by some of its essential consequences, in particular regarding questions of interpretation. Various aspects of the formalism are also reviewed for this purpose.

1 Definition of concepts

The concept of decoherence has become quite popular during the last two decades. However, while its observable consequences have now been clearly confirmed experimentally [1, 2], some misunderstandings regarding its meaning seem to prevail in the literature. The phenomenon itself obviously does not depend on any particular interpretation of quantum theory, but its relevance for them may vary considerably [3, 4]. I am indeed surprised about the indifference of most physicists regarding the potential consequences of decoherence in this respect, since this concept arose as a by-product of arguments favoring either a collapse of the wave function as part of its dynamics, or an Everett-type interpretation. In contrast to the Copenhagen interpretation, which insists on fundamental classical concepts, both these interpretations regard the wave function as a complete and universal representation of reality (cf. [5]).

So let me first emphasize that by decoherence I do neither just mean the disappearance of spatial interference fringes in the statistical distribution of measurement results, nor do I claim that decoherence without additional assumptions is able to solve the infamous measurement problem by *explaining* the stochastic nature of measurements on the basis of a universal Schrödinger equation. Rather, I mean no more (and no less) than the *dynamical dislocalization of quantum mechanical superpositions*, which are defined in an abstract Hilbert space with a local basis – given by particle positions and/or spatial fields, for example. The precise nature of this fundamental Hilbert space basis (the stage for a universal wave function) can only be found in an elusive TOE (theory of everything).

Dislocalization arises through the formation of entanglement of any system under consideration (with states ϕ) with another one or with its unavoidable environment (described by states Φ). This is often achieved by means of a von-Neumann type "measurement" interaction

$$(\sum c_i\phi_i)\Phi_0 \rightarrow \sum c_i\phi_i\Phi_i \quad . \tag{1}$$

Ideal measurements (without recoil or change of the state ϕ_i) define "pure decoherence". Thereafter, these superpositions still *exist*, even though they *are not there* any more [6, 7]. The difference between these two, traditionally equivalent, phrases reflects the essential characteristics of nonlocal quantum reality.

This dislocalization may be reversible ("virtual"), that is, allowing either the complete relocalization of the superposition or its reconstruction (the "quantum erasure" of measurement results), or irreversible in practice ("real" decoherence). The distinction according to the (ir)reversibility of decoherence *explains* also the virtual versus real nature of other "quantum events", such as radioactive decay, particle creation, or excitations. For example, decayed systems remain in a superposition with their undecayed sources until partial waves corresponding to different decay times are decohered from one another. (This gives rise to an exact exponential decay law.) In contrast to complete recoherence (reversal of the dislocalization), quantum erasure is compatible with the irreversible and non-unitary dynamics of open systems – related to a local entropy decrease at the cost of an entropy increase of the environment in classical statistical mechanics [8].

According to (1), dislocalization of superpositions requires a distortion of the environment Φ by the system ϕ rather than a distortion of the system by the environment (such as by classical "noise"). This leads to the important consequence that irreversible decoherence, for example in quantum computers, cannot be error-corrected in the usual manner by means of redundant information storage. Adding extra physical quantum bits to achieve redundancy would in general even raise the quantum computer's vulnerability against decoherence – for the same reason as the increased size of an object normally strengthens its classicality. (Error correction codes proposed in the literature for this purpose are based on the presumption of decoherence-free auxiliary qubits, which may not be very realistic.)

In *special* situations, decoherence is observed as a disappearance of spatial interference fringes. But only for mass points (or center of mass positions of extended objects) are wave functions, which describe superpositions over a configuration space, isomorphic to *spatial* waves, and only *after a position measurement* of many equivalently prepared objects do they form a *statistical distribution.* This is the situation occurring in usual scattering experiments. The interference pattern could then also be obscurred by a slightly varying preparation procedure for the elements of the ensemble (for example due to uncontrollable "noise"), while decoherence affects *individual* quantum states – cf. Equ. (1). Because of the latters' nonlocality it *leads locally* to a reduced density matrix that describes an *apparent ensemble* of states. The conceptually important difference between true and apparent ensembles was clearly pointed out by Bernard d'Espagnat [9] by distinguishing between proper and improper mixtures. In the case of virtual (reversible) decoherence, this difference can be demonstrated operationally by means of recoherence (relocalization of the superposition), which would not be possible for a proper mixture.

Superpositions thus define pure quantum states, which characterize new *individual* properties not present in their components. For example, the superposition of two different spinor states is again an individual spinor state (up or down with respect to another direction); the superposition of a K-meson and its antiparticle defines a new particle (K_{long} or K_{short}); that of a continuum of positions (in the form of a plane wave) defines a certain momentum. Similarly, a superposition of products of the spin states of two particles (even at different places) by means of Clebsch-Gordon coefficients defines an individual state of total spin, while each particle is then in an "improper mixture" because of its virtual decoherence by the other one. Under unitary transformations (described by a Schrödinger equation) these total states remain pure and can never lead to ensembles representing different measurement outcomes in terms of quantum states. However, unitary decoherence may irreversibly lead to local *apparent* ensembles (improper mixtures) which *would* precisely explain the required ensembles of measurement outcomes if they were genuine (proper). This consequence can hardly be an unrelated accident!

2 Roots in nuclear physics

Nuclear physics provides some nice examples of many-particle systems which are nonetheless clearly microscopic (found in energy eigenstates). While I was involved in low energy nuclear physics during the sixties, I became irritated by some methods which were quite successfully used. One of them, called the time-dependent Hartree-Fock approximation, describes "stationary" states of heavy nuclei by means of time-dependent determinants of single-nucleon wave functions. But how can the mathematical solution of an equation $H\psi = E\psi$ know about a concept of time? Similarly, certain nuclei which are intrinsically asymmetric under rotations were often described by means of a time-dependent "cranking model" in order to calculate an effective moment of inertia, or to reproduce a Coriolis type coupling between collective rotational states and individual nucleons. However, both parameters characterize the spectra of energy eigenstates! It turned out that time is here used as a misleading tool to describe *static superpositions* of one-parametric continua of determinants in order to construct quantum states for their corresponding collective degrees of freedom (vibrations or rotations around one axis, for example).

For other collective modes, more than one parameter may be required. General rotations, for example, have to be represented by a non-Abelian symmetry group characterized by three Euler angles. Superpositions then assume the form

$$\Psi = \int d\Omega f(\phi,\theta,\chi) U(\phi,\theta,\chi) \Phi(\mathbf{r}_1,\ldots,\mathbf{r}_n) \quad , \tag{2}$$

where $U(\phi,\theta,\chi)$ is the unitary transformation describing a rotation and $d\Omega$ the volume element in this space, while Φ is a deformed determinant or other "model wave function". There are many other cases where entanglement is circumscribed classically in terms of a time-dependent jargon. Well known is the picture of "vacuum fluctuations", which is used to characterize a static state of entangled quantum fields.

If a variational procedure

$$\delta\langle\Phi|(H-E)|\Phi\rangle = 0 \quad , \tag{3}$$

with determinants Φ consisting of single nucleon wave functions ϕ_i, leads to a deformed solution (as it happens for many heavy nuclei), one must at first conclude that Φ can *not* be an approximation to the correct solution of $H\psi = E\psi$, since it is *not* an angular momentum eigenstate. However, using the degeneracy of these "wrong" solutions under rotations, one may consider their superposition (2) as the next best step. Simultaneous variation of the single-particle wave functions in Φ and the superposition amplitudes $f(\phi,\theta,\chi)$ then leads to angular momentum eigenstates and rotational spectra, including effective Coriolis effects for the single particle motion [10].

The superposition (2) may be regarded as being "dislocalized" over all nucleons in such a way that they are all strongly entangled with one another. A *strong* symmetry violation of the model wave function Φ may be defined by the quasi-orthogonality of slightly different orientations,

$$\langle\Phi|U(\phi,\theta,\chi)|\Phi\rangle \approx 0 \quad \text{for} \quad U \neq 1 \quad , \tag{4}$$

as though the collective orientation were an observable, and $f(\phi,\theta,\chi)$ therefore the corresponding wave function. In a similar way, phonon degrees of freedom arise in solid bodies. Strong violation of rotational symmetry does not require a "needle limit" of strong *geometric* asymmetry: it is a collective effect of many slightly asymmetric single-particle wave functions (subsystems). For product wave functions $\Phi = \prod_i \phi_i(\mathbf{r}_i)$, for example, one would get

$$\langle\Phi|U(\phi,\theta,\chi)|\Phi\rangle = \prod_i \langle\phi_i|U(\phi,\theta,\chi)|\phi_i\rangle =$$
$$\prod_i (1-\epsilon_i) \approx \prod_i \exp(-\epsilon_i) = \exp(-\sum_i \epsilon_i) \quad . \tag{5}$$

This is very similar to decoherence, which is often achieved by means of a product of inner products of many environmental subsystems (such as many scattered particles) [6]. In lowest approximation, each nucleon then "feels" only the deformed (apparently oriented) self-consistent potential produced by the others. While there is no *absolute* orientation in this rotationally symmetric example, the non-relativistic Hamiltonian allows the nucleons in higher order also to experience Coriolis-type consequences in stationary states with non-zero angular momentum eigenvalue.

So one may say that the individual nucleons "observe" an apparent asymmetry in spite of a symmetric global superposition of all orientations. This analogy with a measurement led me to the weird speculation about a nucleus which is big enough to contain a complex subsystem that may resemble a registration device or even a conscious observer. It/he/she would then be entangled with, and thus measure (or be "aware of") a definite orientation of its/his/her "relative world". Does this consequence indicate a way to solve the measurement problem? This picture was also my first attempt towards a (non-relativistic) quantum cosmology – a kind of Everett interpretation as I later discovered. When I learned about the static Wheeler-DeWitt quantum universe described by an equation $H\psi = 0$, it also helped me to understand the concept of time that emerges therein (cf. Sect. 6.2.2 in Ref. [8]). In contrast to a macroscopic body, a nucleus in an energy eigenstate represents a closed quantum system. However, it was absolutely impossible at that time to discuss these ideas with colleagues, or even to present them in a publication.

Macroscopic objects are never found in energy eigenstates, but rather in states of certain (usually time-dependent) orientations or positions. Therefore, it was generally concluded that "quantum theory is not made for macroscopic objects" or even the universe. According to Niels Bohr, macroscopic systems have to be described in terms of *presumed* classical (or "every-day") concepts.

3 The quantum-to-classical transition

Much has been written about the quantum-to-classical transition [11]. It is evidently crucial for a theory that describes reality exclusively in terms of quantum states, while it would be of no more than secondary importance (such as for explaining the absence of interference patterns) if classical concepts were *presumed* for a probabilistic interpretation from the beginning. I could never accept such a fundamental divide between quantum and classical concepts. So one has to understand the different *appearance* of atoms, nuclei and small molecules on the one hand, and macroscopic objects on the other. If both were described quantum mechanically, their energy spectra would differ quantitatively. For example, rotational states of macroscopic objects are very dense. As a consequence, they cannot resist entanglement with their environment even in the case of very weak interactions. Their reduced density matrices must then always represent "mixed states", while the locality of these interactions leads to the vanishing of non-diagonal elements preferentially in the position or "pointer" representation. This is now called decoherence.

Although this term came up more than ten years later (I don't even know who used it first), I pointed out in a number of papers (see [12, 13]) that this disappearance of certain non-diagonal elements of the density matrix explains superselection rules, which were often postulated as restrictions of the superposition principle (for example in axiomatic foundations of quantum theory). They were assumed to hold for specific properties, such as electric charge, as well as for "classical observables", although the axioms did not define a precise boundary between quantum and classical concepts.

In these early papers you will not even find the word "entanglement" – simply because this concept was so rarely used at that time that I did not know this English translation of Schrödinger's *Verschränkung*. So I referred to it as "quantum correlations". Remember that even Schrödinger, in his famous paper of 1935 [14], regarded *Verschränkung* as a mysterious probability relation (which would have to characterize ensembles rather than individual states), since he was convinced that reality has to be defined in space and time.

However, what I had in mind went beyond what is now called decoherence, since it was inspired by the above mentioned picture of an observer inside a closed quantum system. An external observer, who is *part of the environment* of the observed object, becomes entangled, too, with the property he is observing – just as the internal observer is entangled with the orientation of the deformed nucleus. He is thus part of a much bigger "nucleus" (or closed system): the quantum universe. So he "feels" or can be aware only of a definite value of this property (or separately of different values in different "Everett worlds"). All you have to assume is that his *various* quantum states which do exist as factor states in these *components* of the global wave function are the true carriers of awareness. This is even plausible from a quite conventional point of view, since these component states, which are a consequence of the Schrödinger equation, possess all properties required to define observers, such as complexity and dynamical stability (memory). Indeed, these states are the same ones that would arise in appropriate collapse theories, which eliminate all but one components from reality by means of a modification of the dynamical law. I do not see why such a modification should be required. A genuine collapse that was simply *triggered* by irreversible decoherence (as recently suggested in a very clever way by Roland Omnès [15]) would not lead to any observable consequences. It may then be just a matter of taste whether you apply Occam's razor to the facts (by inventing new dynamical laws to cut off what you cannot see) or to the laws (by leaving the Schrödinger equation unchanged) – although this choice must clearly have *cosmological* consequences (such as the possibility of a symmetric superposition of very many symmetry violating "worlds").

For me the most important fruit of decoherence (that is, of a universal entanglement) is the fact that no classical concepts are required any more on a fundamental level. There is then also no need for a fundamental concept of "observables" (which would assume *certain* values only upon measurement) – see Chap. 4, or for uncertainty relations restricting such values: the Fourier theorem for the wave function explains this "uncertainty" in a natural way – well known for classical radio waves, which are themselves real and certain. When Bohr and Heisenberg insisted that the uncertainty relations go beyond the Fourier theorem, they were apparently thinking of *spatial* wave functions only (thus neglecting entanglement).

For microscopic objects, which can be sufficiently isolated, the experimental physicist has a choice between mutually exclusive ("conjugate") measurements, while macroscopic properties are decohered by their unavoidable environment in a *general and specific* manner. This explains their classical appearance. The corresponding quasi-classical basis in Hilbert space then appears as a classical configuration space, while the conventional "quantization" procedure may be regarded as the re-introduction of these lost superpositions into the (approximately valid) classical theory.

Let me here quote from a recent publication by Ulfbeck and Aage Bohr [16] from Copenhagen regarding the nature of *quantum events*: "No event takes place in the source itself as a precursor of the click in the counter ...". Hence, there is no decay event in the atom, for example! So far I agree; this conclusion, which is in contrast to earlier interpretations of quantum theory, is required by experiments which use reflected decay fragments to demonstrate recoherence (state vector revival) or interference with partial waves resulting from later decay times. In order to appreciate this important change in the Copenhagen interpretation, one may compare the *new* version with Pauli's claim that "the appearance of a certain position or momentum *of a particle* is a creation outside the laws of nature" (my italics). However, Ulfbeck and Bohr continue their sentence of above: "... where the wave function loses its meaning." Here I strongly disagree. After all, it is precisely the arising entanglement with the environment, described by a global wave function, which explains decoherence. These authors are correct, though, when placing the creation of stochastic "events" in the apparatus, where the dislocalization of the relevant superposition becomes irreversible FAPP (for all practical purposes), thus creating an *apparent* ensemble of quasi-classical wave packets. The dishonesty of the Copenhagen interpretation consists in switching concepts on demand and regarding the (genuine or apparent) collapse as a "normal increase of information" – as though the wave function represented no more than an ensemble of *possible* states.

Of course, you may *pragmatically* use classical concepts as though they were fundamental – even when studying decoherence as a phenomenon. One cannot expect the practicing physicist always to argue in terms of a universal wave function. But he may keep in mind that there *is* a consistent description (thus representing a "quantum reality") underlying his classical terminology. Similarly, a high energy physicist uses the concepts of momentum and energy (rather than relativistic "momenergy") to describe the objects in his laboratory. Fortunately, there are other fruits of decoherence in the form of observable phenomena which demonstrate decoherence in action [1, 2]. However, the derivability of classical (such as particle) concepts undermines any motivation for the Heisenberg picture as well as for Bohm's quantum mechanics.

4 Quantum mechanics without observables

[1]

In quantum theory, measurements are traditionally described by means of "observables", which are in the Heisenberg picture assumed to replace the classical *variables*, and therefore to carry the dynamical time dependence. They are formally represented by hermitean operators, and introduced in addition to the concepts of quantum states and their dynamics as a fundamental and independent ingredient of quantum theory. However, even though often forming the starting point of a formal quantization procedure, this ingredient may not be separately required if physical states are universally described by general quantum states (superpositions in an appropriate basis) and their dynamics. This interpretation, to be further explained below, would comply with John Bell's quest for a theory in terms of "beables" rather than observables [17]. It was for this reason that his preference shifted from Bohm's theory to collapse models (where wave functions are assumed to completely describe *reality*) during his last years. (Another reason was his antipathy against

[1] This chapter is based on Sect. 2.2 of [7]. It may be omitted for a quick reading.

the "extravagance" – as he called it – of the multiplicity of worlds, which appears in the form of myriads of empty components as well in Bohm's never collapsing wave function.)

Let $|\alpha\rangle$ be an arbitrary quantum state (perhaps experimentally prepared by means of a "filter" – see below). The *phenomenological* probability for finding the system in another quantum state $|n\rangle$, say, after an appropriate measurement, is given by means of their inner product, $p_n = |\langle n \mid \alpha\rangle|^2$, where both state vectors are assumed to be normalized. This may either correspond to a collapse or a branching of the wave function – though neglecting the state of the apparatus and environment. The state $|n\rangle$ represents here a specific measurement. In a position measurement, for example, the number n has to be replaced with the continuous coordinates x, y, z, leading to the "improper" Hilbert states $|\mathbf{r}\rangle$. Measurements are called "of the first kind" or "ideal" if the system will again be found in the state $|n\rangle$ (except for a phase factor) whenever the measurement is immediately repeated. *Preparations* of states can be regarded as measurements which *select* a certain subset of outcomes for further measurements. n-preparations are therefore also called n-filters, since all "not-n" results are thereby excluded from the subsequent experiment proper. The above probabilities can be written in the form $p_n = \langle\alpha \mid P_n \mid \alpha\rangle$, with a special "observable" $P_n := |n\rangle\langle n|$, which is thus *derived* from the kinematical concept of quantum *states*, and not introduced as a fundamental concept.

Instead of these special "n or not-n measurements" (with fixed n), one can also perform more general "n_1 or n_2 or ... measurements", with all n_i's mutually exclusive ($\langle n_i|n_j\rangle = \delta_{ij}$). If the states forming such a set $\{|n\rangle\}$ are pure and exhaustive (that is, complete, $\sum P_n = 1$), they represent a basis of the corresponding Hilbert space. By introducing an arbitrary "measurement scale" a_n, one may construct *general* observables

$$A = \sum |n\rangle a_n \langle n| \quad , \tag{6}$$

which permit the definition of "expectation values"

$$\langle\alpha \mid A \mid \alpha\rangle = \sum p_n a_n \quad . \tag{7}$$

In the special case of a yes-no measurement, one has $a_n = \delta_{nn_0}$, and expectation values become probabilities. Finding the state $|n\rangle$ during a measurement is then also expressed as "finding the value a_n of an observable". A uniquely invertible change of scale, $b_n = f(a_n)$, describes the *same* physical measurement; for position measurements of a particle it would simply represent a coordinate transformation. Even a measurement of the particle's potential energy is equivalent to a position measurement (up to degeneracy) if the function $V(\mathbf{r})$ is *given*.

According to this definition, quantum expectation values must not be understood as mean values in an ensemble that represents ignorance of the precise state. Rather, they have to be interpreted as probabilities for *potentially arising* quantum states $|n\rangle$ – regardless of the interpretation of this stochastic process. If the set $\{|n\rangle\}$ of such potential states forms a basis, any state $|\alpha\rangle$ can be represented as a superposition $|\alpha\rangle = \sum c_n|n\rangle$. In general, it neither forms an n_0-state nor any not-n_0 state. Its dependence on the complex coefficients c_n requires that states which differ from one another by a numerical factor must be different "in reality". This is true even though they represent the same "ray" in Hilbert space and cannot, according to the measurement postulate, be distinguished operationally. The states $|n_1\rangle + |n_2\rangle$ and $|n_1\rangle - |n_2\rangle$ could not be physically different from another if $|n_2\rangle$ and $-|n_2\rangle$ were really the *same* state. While operationally meaningless in the state $|n_2\rangle$ by itself, any numerical factor would become relevant in the case of *recoherence*. (Only a *global* factor would be "redundant".) For this reason, projection operators $|n\rangle\langle n|$ are insufficient to characterize quantum states.

The expansion coefficients c_n, relating physically meaningful states – for example those describing different spin directions or different versions of the K-meson – must in principle be determined (relative to one another) by appropriate experiments. However, they can often be derived from a previously known (or conjectured) classical theory by means of "quantization rules". In this case, the classical configurations q (such as particle positions or field variables) are *postulated* to parametrize a basis in Hilbert space, $\{|q\rangle\}$, while the canonical momenta p parametrize another one, $\{|p\rangle\}$. Their corresponding observables,

$$Q = \int dq\, |q\rangle q \langle q| \quad \text{and} \quad \mathrm{P} = \int \mathrm{dp}\, |\mathrm{p}\rangle \mathrm{p} \langle \mathrm{p}| \quad , \tag{8}$$

are required to obey commutation relations in analogy to the classical Poisson brackets. In this way, they form an important *tool* for constructing and interpreting the specific Hilbert space of quantum states. These commutators essentially determine the unitary transformation $\langle p \mid q\rangle$ (e.g. as a Fourier transform $\mathrm{e}^{\mathrm{ipq}}$) – thus more than what could be defined by means of the projection operators $|q\rangle\langle q|$ and $|p\rangle\langle p|$. This algebraic procedure is mathematically very elegant and appealing, since the Poisson brackets and commutators may represent generalized symmetry transformations. However, the *concept* of observables (which form the algebra) can be derived from the more fundamental one of state vectors and their inner products, as described above.

Physical states are assumed to vary in time in accordance with a dynamical law – in quantum mechanics of the form $\mathrm{i}\partial_\mathrm{t}|\alpha\rangle = \mathrm{H}|\alpha\rangle$. In contrast, a measurement device is usually defined regardless of time. This must then also hold for the observable representing it, or for its eigenbasis $\{|n\rangle\}$. The probabilities $p_n(t) = |\langle n \mid \alpha(t)\rangle|^2$ will therefore vary with time according to the time-dependence of the physical states $|\alpha\rangle$. It is well known that this (Schrödinger) time dependence is formally equivalent to the (inverse) time dependence of observables (or the reference states $|n\rangle$). Since observables "correspond" to classical *variables*, this time dependence appeared suggestive in the Heisenberg–Born–Jordan algebraic approach to quantum theory. However, the absence of *dynamical states* $|\alpha(t)\rangle$ from this Heisenberg picture, a consequence of insisting on *classical* kinematical concepts, leads to paradoxes and conceptual inconsistencies (complementarity, dualism, quantum logic, quantum information, and all that).

An environment-induced superselection rule means that certain superpositions are highly unstable against decoherence. It is then impossible in practice to construct measurement devices for them. This *empirical* situation has led some physicists to *deny the existence* of these superpositions and their corresponding observables – either by postulate or by formal manipulations of dubious interpretation, often including infinities or non-separable Hilbert spaces.

While *any* basis $\{|n\rangle\}$ in Hilbert space defines formal probabilities, $p_n = |\langle n|\alpha\rangle|^2$, only a basis consisting of states that are not immediately destroyed by decoherence defines "realizable observables". Since the latter usually form a genuine subset of *all* formal observables (diagonalizable operators), they must contain a nontrivial "center" in algebraic terms. It consists of those which commute with all the rest. Observables forming the center may be regarded as "classical", since they can be measured simultaneously with *all* realizable ones. In the algebraic approach to quantum theory, this center appears as part of its axiomatic structure [18]. However, since the condition of decoherence has to be considered quantitatively (and may even vary to some extent with the specific nature of the environment), this algebraic classification remains an approximate and dynamically emerging scheme.

These "classical" observables thus characterize the subspaces into which superpositions decohere. Hence, even if the superposition of a right-handed and a left-handed chiral molecule, say, *could* be prepared by means of an appropriate (very fast) measurement of the first kind, it would be destroyed before the measurement may be repeated for a test. In contrast, the chiral states of all individual molecules in a bag of sugar are "robust" in a normal environment, and thus retain this property *individually* over time intervals which by far exceed thermal relaxation times. This stability may even be increased by the quantum Zeno effect (see [19] for a consistent and exhaustive discussion). Therefore, chirality does not only appear classical in these cases, but also as an approximate constant of the motion that has to be taken into account for defining canonical ensembles in thermodynamics.

The above-used description of measurements of the first kind by means of probabilities for transitions $|\alpha\rangle \to |n\rangle$ (or, for that matter, by corresponding observables) is phenomenological. However, measurements should be described *dynamically* as interactions between the measured system and the measurement device. The observable (that is, the measurement basis) should thus be derived from the corresponding interaction Hamiltonian and the initial state of the device. As shown by von Neumann, this interaction must be diagonal with respect to the measurement basis (see also [20]). Its diagonal matrix elements are operators which act on the quantum state of the device in such a way that the "pointer" moves into a position appropriate for being read, $|n\rangle|\Phi_0\rangle \to |n\rangle|\Phi_n\rangle$. Here, the first ket refers to the system, the second one to the device. The states $|\Phi_n\rangle$, representing different pointer positions, must approximately be mutually orthogonal, and "classical" in the explained sense.

Because of the dynamical superposition principle, an initial superposition $\sum c_n|n\rangle$ does *not* lead to definite pointer positions (with their empirically observed frequencies). If decoherence is neglected, one obtains their *entangled superposition* $\sum c_n|n\rangle|\Phi_n\rangle$, that is, a state that is different from all potential measurement outcomes $|n\rangle|\Phi_n\rangle$. This dilemma represents the "quantum measurement problem". Von Neumann's interaction is nonetheless regarded as the first step of a measurement (a "pre-measurement"). Yet, a collapse seems to be required – now in the measurement device rather than in the microscopic system. Because of the entanglement between system and apparatus, it would then affect the total system. (Some authors seem to have taken the phenomenological collapse in the *microscopic system* by itself too literally, and therefore disregarded the state of the measurement device in their measurement theory. Such an approach is based on the assumption that quantum states must always exist for all systems. This would be in conflict with quantum nonlocality, even though it may be in accordance with early interpretations of the quantum formalism.)

If, in a certain measurement, a whole subset of states $|n\rangle$ leads to the same pointer position $|\Phi_{n_0}\rangle$, these states can not be distinguished by this measurement. According to von Neumann's interaction, the pointer state $|\Phi_{n_0}\rangle$ will now be correlated with the *projection* of the initial state onto the subspace spanned by this subset. A corresponding *collapse* was therefore postulated by Lüders [21] as a generalization of von Neumann's "first intervention" (as he called the collapse dynamics).

In this sense, the interaction with an appropriate measuring device *defines* an observable. The formal time dependence of observables according to the Heisenberg picture would then describe a time dependence of the states diagonalizing the interaction Hamiltonian, such that, paradoxically, the device would be assumed to be dynamically controlled by the Hamiltonian of the system.

The question whether a certain formal observable (that is, a diagonalizable operator) can be *physically realized* can only be answered by taking into account the unavoidable environment. A macroscopic measurement device is *always* asssumed to decohere into its macroscopic pointer states. However, as mentioned in Chapter 3, environment-induced decoherence by itself does not solve the measurement problem, since the "pointer states" $|\Phi_n\rangle$ may be defined to include the total environment (the "rest of the world"). Identifying the thus arising global superposition with an *ensemble* of states (represented by a statistical operator ρ) that leads to the same expectation values $\langle A\rangle = \mathrm{tr}(\mathrm{A}\rho)$ for a *limited* set of observables $\{A\}$ would beg the question. This merely operational argument is nonetheless often found in the literature.

5 Rules versus tools

As the Everett interpretation describes a "branching quantum world", which mimics a collapsing wave function to the internal observer, the question is often raised for the precise *rules* of this branching – similar to the dynamical rules for a collapse. Such collapse rules would have to define the individual branches (or the "pointer states") as well as their dynamical probabilities. In contrast, decoherence describes the branching by means of the Schrödinger equation as a dislocalization of initially local superpositions in such a way that the latter become gradually inaccessible to any local observer. Decoherence neither defines nor explains this ultimate (conscious) observer. While the branching is ultimately justified by the observer's locality, the dislocalization is an *objective* dynamical process – in particular occurring in measurement devices.

This unitary dynamical process causes the non-diagonal elements of the reduced density matrices of all dynamically involved local systems (such as those forming a chain of interactions which lead to an observation) to gradually vanish. These *indicators* of dislocalized superpositions are therefore often used to *define* decoherence. However, subsystems and their density matrices are no more than convenient conceptual tools, useful because of the locality of all interactions and the causal structure of our world (based on cosmic initial conditions that are responsible for the arrow of time [8]). In contradistinction to a nonlocal superposition, the concept of a density matrix *presumes* the probability interpretation. The degree of diagonalization of the reduced density matrices would depend on the precise choice and boundaries of subsystems, but this is irrelevant for a sufficient definition of "macroscopically distinct" global branches FAPP. Decoherence may thus be called a *collapse without a collapse*.

While a genuine collapse theory would have to postulate (as part of the dynamical law) probabilities for its various *possible* outcomes, in an Everett world *all* branches are assumed to remain in existence. We can then meaningfully argue only about *frequencies* of outcomes (such as spots on a screen) in *series* of measurements that are performed in our branch. Graham was able to show more than thirty years ago [22] that all those very abundant (by number) "maverick Everett worlds" which do not possess frequencies in accordance with the Born probabilities possess a norm that vanishes with increasing size of the series. While their exclusion is nonetheless *not* a trivial assumption, the norm plays here a similar rôle as phase space does in classical statistical physics: it is dynamically conserved under the Schrödinger equation, and thus an appropriate measure of probability.

6 Nonlocality

Let me continue with another reminiscence from the "dark ages of decoherence" (that ended not before Wojcziech Zurek had published his first papers on the subject in the Physical Review [20]). After I had completed the manuscript for my first paper on what is now called decoherence, the only well known physicist who responded to it in a positive way for a long time was Eugene Wigner. He helped me to get it published, and he also arranged for an invitation to a conference on the foundations of quantum theory to be held in Varenna in 1970, organized by Bernard d'Espagnat [23].

When I arrived at Varenna, I found the participants (John Bell included) in hot debates about the first experimental results regarding the Bell inequalities, which had been published a few years before this conference [24]. I had never heard of them, but I could not quite share the general excitement, since I was already entirely convinced that entanglement (and hence nonlocality) was a well founded property of quantum states, which in my opinion described reality rather than probability correlations. So I concluded that everybody would now soon agree.

Obviously I was far too optimistic. Some physicists are searching for loopholes in the experiments which confirm the violation of these inequalities until today – even though all experimental results were precisely predicted by quantum theory. Others (perhaps still the majority) are interpreting nonlocality as a "spooky action at a distance", which would have to affect tacitly presumed local quantities (such as described by classical concepts). I cannot see anything but prejudice (once shared by Einstein and Schrödinger!) in such an assumption about reality. It is amazing that even Bohm, who did assume the nonlocal wave function to be real, added classical concepts to describe another (local) reality, which would have to include the observer, and for which the wave function acts as no more than a pilot wave.

It appears strange, too, that certain "measures of entanglement" that have recently been much in use [25] measure only reversible or usable entanglement, while quite incorrectly regarding irreversible entanglement (decoherence) as "noise" or "distortion". It is certainly not an accident that this position appears related to Ulfbeck and Bohr's above-mentioned statement. The observable consequences of Equ. (1) demonstrate that quantum measurements can *not* be regarded as describing a "mere increase of information" – even in the absence of any recoil. Quantum measurements produce *real* nonlocal entanglement.

If reality is accepted to be *kinematically* nonlocal, you also don't need any "spooky teleportation" in order to explain certain experiments that appear particularly attractive to science fiction authors. In all these experiments you have to *prepare in advance* a nonlocal (entangled) state that contains, in one of its components, precisely what is later claimed to be ported already at its final position. For example, in such a setting two spinors have to be prepared in the form of a Bell state

$$| \uparrow\rangle_A | \downarrow\rangle_B - | \downarrow\rangle_A | \uparrow\rangle_B = \\ | \rightarrow\rangle_A | \leftarrow\rangle_B - | \leftarrow\rangle_A | \rightarrow\rangle_B = \dots \quad , \tag{9}$$

where A and B refer to Alice's and Bob's place, respectively. Nothing has to be ported any more when Alice, say, performs a measurement – for example that of another (local) Bell state that includes her spinor of (9). Because of the "real" (irreversible) decoherence of the nonlocal superposition caused by this measurement, the *initial* Bell state becomes an apparent ensemble, such that the entanglement *appears* to be a statistical correlation from the point of view of all local observers (such as Bob). His apparently incomplete information may then be "completed" by apparently classical means (Alice sending a message to Bob). In quantum terms, this "information transfer" means that Bob, too, becomes a consistent member of the (partly irreversible) global entanglement. This is experienced by him (in all his branches) as a collapse of the wave function (see Joos's Sect. 3.4.2 of [7]). Alice assumes here the rôle of Wigner's well known "friend". If Pauli's remark of Chap. 5 were right, though, something like telekinesis would indeed have to occur ("outside the laws of nature"). The term "quantum information" instead of entanglement is therefore quite misleading: entanglement is part of quantum reality – even though it may become indistinguishable from a statistical correlation in practice.

You would need a similar initial Bell-type superposition of the kind

$$|CK\rangle_A |noCK\rangle_B - |noCK\rangle_A |CK\rangle_B \tag{10}$$

in order to "beam" Captain Kirk (CK) from Alice's to Bob's place, provided he *could* be shielded against decoherence until Alice "measures" his absence at her place. (This hypothetical isolation would require the existence of a local Schrödinger cat state $|CK\rangle \pm |noCK\rangle$.) However, the beamed Captain Kirk could not be one who knows what happened at Alice's place *after* preparation of the initial Bell state. You would need a tremendously more complex entangled state, that had to contain *all possibilities* as part of its nonlocal quantum reality, in order to be able to decide later *what* to beam. The term "quantum teleportation" drastically misleads and should in my opinion not be used by serious scientists.

7 Information loss (paradox?)

The collapse of the wave function (without observing the outcome) or any other *indeterministic* process would represent a dynamical information loss, since a pure state is transformed into an ensemble of possible states (described by a proper mixture, for example). The dislocalization of quantum mechanical superpositions, on the other hand, leads to an *apparent* information loss, since the relevant phase relations merely become irrelevant for all practical purposes of local observers. I will now argue that the "information loss paradox of black holes" (Hawking's lost bet) is precisely based on this decoherence (or otherwise on the collapse of the wave function), and *not* a specific property of black holes.

For a better understanding one may first consider irreversible processes in classical mechanics, such as Boltzmann's molecular collisions. Since they are based on deterministic dynamics, *ensemble entropy* is here conserved (in analogy to quantum unitarity). However, collisions lead to the formation of uncontrollable statistical correlations, which are irrelevant for all practical purposes in the future. (They are important, though, for the correct backward dynamics because of the specific cosmic initial condition that has to be assumed for this Universe.) This apparent loss (namely, the dislocalization) of *information* in this classical case affects *physical entropy*, since this entropy concept disregards *by definition* the arising uncontrollable correlations [8]. It is defined as an extensive (additive) quantity, usually in terms of "representative ensembles" characterizing the local macroscopic variables, while microscopic (the *real*) states – including those of subsystems – remain objectively determined in principle by the global initial conditions because of the presumed classical mechanical laws. In contrast, quantum mechanical subsystems possess non-vanishing objective entropy (described by improper mixtures) even for a completely defined global state.

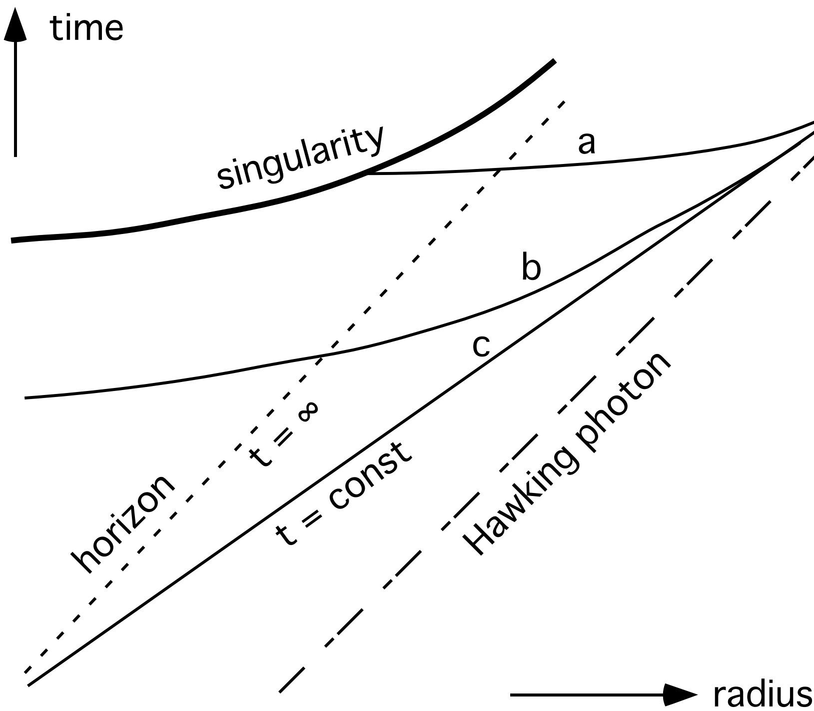

Figure 1: Various simultaneities for a spherical black hole in a Kruskal type diagram: (a) hitting the singularity, (b) entering the regular interior region only, (c) completely remaining outside (Schwarzschild time coordinate t). Light cones open everywhere at $\pm 45^0$ around the vertical time axis in this diagram, while lengths are distorted. Schwarzschild time is appropriate in particular for posing external boundary conditions. The angle between the horizon and the line $t = const$ can here be arbitrarily changed by a passive time translation. This includes the (apparently close) vicinity of the horizon, which can thus be arbitrarily "blown up" in the diagram – thus transforming *any* Schwarzschild time into the horizontal line $t = 0$, for example.

In general relativity (GR), "information" may disappear when physical objects fall onto a spacetime singularity, but in classical physics the real state of external matter would exist and remain well defined. For quantum mechanics on a classical spacetime, the information loss would have to include all existing entanglement with external matter, thus transforming the latter's improper mixture into a proper one. This conclusion seems to remain true when the black hole disappears by means of Hawking radiation, and this has been regarded as a paradox, since it would violate unitarity.

One may consider the spacetime geometry of a black hole in Kruskal-type coordinates (see Figure 1). Simultaneities used by external observers in asymptotic Minkowski spacetime (such as time coordinates in the black hole's rest system) can here be continued in space towards the center of the spherical black hole in different ways. If everywhere chosen according to the Schwarzschild time coordinate t, for example, they would never intersect the horizon, but this choice does *not* affect the density matrix representing the region far from the horizon (far right in the Figure). The information loss noticed by an external observer can therefore not be *caused* by the singularity – no matter how long he waits. Not even the horizon ever enters his past, and thus never becomes a "fact" for him, while the Hawking radiation which he may observe would originate earlier in Schwarzschild time than the horizon. The close vicinity of the horizon can causally affect only the very distant future.

On the other hand, a macroscopic black hole is permanently affected by various kinds of decoherence [26] – most importantly by means of its retarded radiation. So this quantum radiation must be highly entangled with the remaining black hole, and therefore with all radiation that is emitted later [27]. If usable information about the black hole is stored in the external world (such as in the form of emitted light), it defines separate Everett branches. While the unitary dynamics determines the later global quantum state uniquely, it does *not* determine an observer's branch: the present state of an observer will have *many* successors in the future. Any confirmation of the black hole's unitary dynamics would thus require the recovery of *all* coherence, including the recombination of Everett worlds – just as it would be required to confirm unitarity for all other macroscopic objects. In practice, their evolution is irreversible. This means that the answer to Hawking's bet has nothing specifically to do with black holes [28].

The spacetime metric with its event horizons and singularities is "real and certain" only in classical GR. In quantum gravity, even the spacetime geometry on which simultaneities are to be defined has to be replaced by an entangled state of matter and geometry. *All* macroscopic properties are thereby decohered and have to be associated with different (and further branching) Everett worlds. The Wheeler-DeWitt wave function $\Psi[{}^3G, \phi_{matter}]$ (or its generalization to unified theories), which describes their global superposition, has to obey certain boundary conditions. For example, it may have to exclude singularities, or just any entanglement between them and regular regions. This would strongly affect the wave function on all spatial geometries which contain a black hole horizon. Here, the WKB approximation, which allows quasi-classical spacetime (hence time) to *emerge* by means of the process of decoherence, may completely break down [29], while the classical spacetime diagram of Figure 1 would lose its meaning close to the horizon.

8 Dynamics of entanglement

The entangled state of any two quantum systems, if assumed to be pure, can always be written as a single sum in the *Schmidt canonical form* [30]

$$\psi = \sum_i \sqrt{p_i}\phi_i\Phi_i \quad , \tag{11}$$

where the states ϕ_i and Φ_i forming the two bases are *determined* (up to linear combinations between degenerate coefficients) by the total state ψ. The coefficients can be chosen real and positive by an appropriate choice of phases for the states forming the Schmidt bases, and have therefore been written in the form $\sqrt{p_i}$. In contrast to Equ. (1), the states Φ_i are now assumed to be orthogonal: the expansion (11) is thus in general different from (1). This Schmidt representation is equivalent to the diagonal form of the reduced density matrices

$$\begin{aligned} \rho_\phi &= \sum_i |\phi_i\rangle p_i \langle\phi_i| \quad , \\ \rho_\Phi &= \sum_i |\Phi_i\rangle p_i \langle\Phi_i| \quad . \end{aligned} \tag{12}$$

Since all systems must be assumed to be entangled with their environments, the "second" system has in principle always to be understood as the "rest of the universe" in order to represent a realistic situation.

If the total state ψ depends on time, the bases ϕ_i and Φ_i *and* the coefficients $\sqrt{p_i}$ must carry a separate time dependence, which is determined, too, by that of the global state $\psi(t)$. It can be explicitly described [13] by

$$\begin{aligned} \frac{d\sqrt{p_i}}{dt} &= Im \sum_j \sqrt{p_i}\ \langle ii|H|jj\rangle \\ \mathrm{i}\frac{d\phi_i}{dt} &= \sum_{j\neq i} (p_i - p_j)^{-1} \sum_m \sqrt{p_m}\ \left[\ \sqrt{p_i}\ \langle ji|H|mm\rangle - \sqrt{p_i}\ \langle mm|H|ij\rangle\ \right]\ \phi_i \\ \mathrm{i}\frac{d\Phi_i}{dt} &= \sum_{j\neq i} (p_i - p_j)^{-1} \sum_m \sqrt{p_m}\ \left[\ \sqrt{p_i}\ \langle ij|H|mm\rangle - \sqrt{p_i}\ \langle mm|H|ji\rangle\ \right]\ \Phi_j \\ &+ \sqrt{p_i}\ Re \sum_m \sqrt{p_m}\ \langle ii|H|mm\rangle\ \Phi_i \quad . \end{aligned} \tag{13}$$

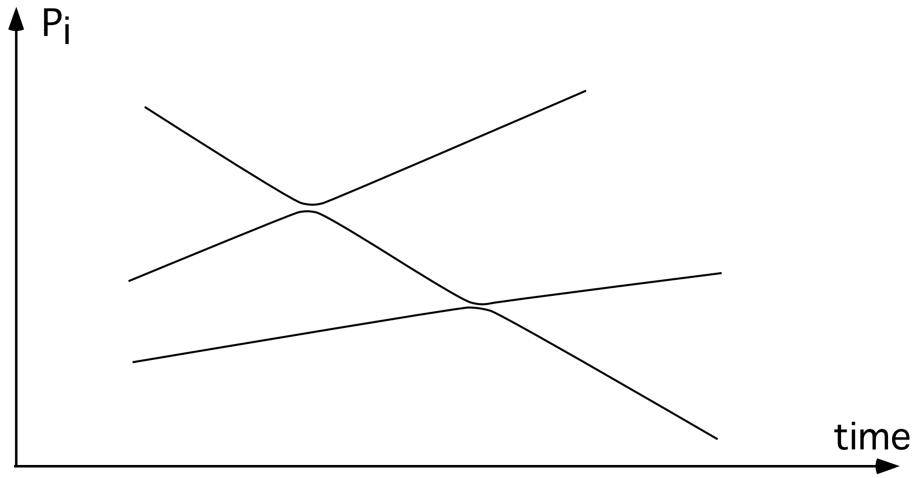

Figure 2: Trajectories of different probabilities $p_i(t)$ repel each other, while their corresponding factorizing Schmidt components interchange their identity (including all memories). Causal histories of Schmidt components thus intersect in this diagram, although they never touch.

The asymmetry between the two subsystems described by ϕ and Φ is here due to an asymmetric phase choice. It could be avoided by using a different phase convention [31].

In classical physics, subsystems would evolve deterministically, controlled by time-dependent Hamiltonians depending on the state of the other system (thus forming coupled deterministic dynamics). This classical picture of time-dependent Hamiltonians is often, not very consistently, used also in quantum mechanics – for example in the form of perturbing "kicks" instead of genuine quantum interactions. In contrast, Equs. (13) define highly nontrivial (not practically usable) nonunitary subsystem dynamics. For this reason, the "probabilities" p_i and the entropy $\sum p_i \ln p_i$ defined by them must usually change in time. In particular, initially separating systems will become entangled.

Although these equations define a continuous evolution separately for each term of the Schmidt representation, this dynamics seems to be singular whenever two diagonal elements p_i of the density matrix become equal. However, closer inspection of the dynamics reveals that two eigenvalues coming close repel each other (unless the corresponding matrix elements of the Hamiltonian vanish exactly), and therefore never intersect as functions of time (see Figure 2). Thereby, the quasi-singular dynamics (13) of the states forces the latter to interchange their identity within a very short time. In other words, degeneracy of probatilities can be assumed never to occur, while the formal continuity of Schmidt components is entirely unphysical (not representing preserved memory). Subsystem density matrices are *not* affected by this phenomenon, since the resonance terms are a consequence of the ambiguity of their degenerate eigenstates. The non-unitary dynamics of entangled density matrices can implicitly (that is, depending on the solutions of (13)) be written [32]

$$\begin{aligned} i\frac{d\rho_\Phi}{dt} \qquad &:= i\frac{d\sum p_i\Phi_i\Phi_i^*}{dt} \\ &= \sum_{i,j}\left(\sqrt{p_i}\langle ij|H|\psi\rangle - \sqrt{p_j}\langle\psi|H|ji\rangle\right)\Phi_i\Phi_j^* \quad . \end{aligned} \tag{14}$$

Of special interest for the concept of decoherence are initially separating (factorizing) states. While this assumption enforces an initial degeneracy to exist between all vanishing probabilities, the initial component with $p_0 = 1$ must at least quadratically depend on time because of the time reversal symmetry of the global Schrödinger equation. In this small-times approximation its precise form can be derived by means of perturbation theory as

$$p_0(t) \approx 1 - t^2 A \tag{15}$$

where the quantity

$$A = \sum_{j\neq 0, m\neq 0} |\langle jm|H|00\rangle|^2 \quad , \tag{16}$$

has been called a *deseparation parameter*. It measures the arising entanglement (that is, the growing deviation from separating states). Index pairs jm here refer to product states $\phi_j\Phi_m$. Note that A is different from the quantity

$$B = \sum_{jm\neq 00} |\langle jm|H|00\rangle|^2 \geq A \quad , \tag{17}$$

which measures the *total* change of the global state in this approximation (including the "classical" change characterizing a time-dependent product). If the environment and the interaction Hamiltonian H are given, the deseparation parameter A can be used to estimate the robustness of certain states against decoherence. For example, coupled harmonic oscillators turn out to be robust when in coherent states (such as in states describing classical fields), while their energy eigenstates (such as photon number eigenstates) are unstable [13].

9 Concluding remarks

To conclude, let me emphasize that the concept of decoherence does not contain any new physics beyond the established framework of quantum theory. Rather, it is is a consequence of the universal application of quantum concepts (superpositions) and their unitary dynamics.

However, a consistent interpretation of this theory in accordance with the observed world requires a *novel and nontrivial identification of observers* with appropriate quantum states of local systems which exist only in certain, dynamically autonomous *components* of the global wave function. Accordingly, it is the observer who "splits" indeterministically – not the (quantum) world.

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